9. Endogenous age structure in descriptive macroeconomic growth models: a general framework and some steady state analysis*

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9.1. INTRODUCTION

Since Solow's (1956) contribution, continuous time descriptive macroeconomic growth models have proved to be not only cornerstone contributions to economic growth theory but also, once amended to include realistic refinements such as human capital, very useful tools for the empirical analysis of growth (Barro and Sala-i-Martin, 1995; Mankiw et al., 1992 and subsequent contributions). Another fundamental descriptive model is Goodwin's (1967) growth-cycle, which represents the paradigm of non-linear dynamics in economics.

In descriptive macroeconomic growth models population plays an important role – through the growth of the labour supply it triggers the growth of absolute output – yet it is taken as fully exogenous to the economic system. Despite the large number of papers aiming to endogenise population in macroeconomic growth models, the main feature of the demographic system, i.e. age structure, has, with few exceptions, been neglected or dealt with in an oversimplified manner, as for instance in Overlapping Generation Models. This is a somewhat odd fact given that, at least from Malthus (1798) onwards, the role of age structure as the 'conveyor belt' linking economic shocks, their demographic response, for instance in terms of fertility, and their ultimate feedback on the economic system, was well present to classical economists. For instance, McCulloch (1854, p. 34) stated 'the supply of labourers in the market can neither be speedily increased when wages rise, nor speedily diminished when they fall. When wages rise a period of eighteen or twenty years must elapse before the stimulus, given the principle of population, can be felt in the market'. McCulloch's description provides the first definition of the causal factor underlying the fundamental demo-economic concept of Malthusian cycles, which are the consequence: '...of the lags between the response of fertility to current labour market conditions and the time when the resulting births actually enter the labour force' (Lee 1997, p. 1097). Malthusian cycles are a major example of demo-economic oscillations and a well documented fact in population economics. See for instance the oscillations of birth, death and wage rates in UK history (Lee and Anderson, 2002, p. 201), the Swedish cycles 1700–1914, and the post-transitional fertility fluctuations, such as the baby-booms observed in the Western world in the 1960s, sometimes explained by Easterlin's effect (Easterlin, 1978).

There are some exceptions to the paucity of theoretical studies of the dynamic interplay between the population age structure and macro-dynamics. Several papers have been devoted to the search for Easterlinian mechanisms leading to sustained Malthusian oscillations (Lee, 1974; Samuelson, 1976; Frauenthal and Swick, 1983; Feichtinger and Sorger, 1989 and 1990; Feichtinger and Dockner, 1990; Chu and Lu, 1995). Most such works, however, are simplistic in that they include the economic system only implicitly through some non-linear demographic relationship. Manfredi and Fanti (2006a; 2006b) made a more genuine attempt to integrate the age structure of the population and the macroeconomic structure, within Goodwin's model. In order to end up with a tractable model (i.e. described by ordinary differential equations, ODEs) they used a simplified representation of age structure by considering three additional differential equations modelling the main phases of the individual life cycle, namely youth, adulthood, and retirement. The ensuing model shows a clear-cut dynamic effect of age structure on growth, particularly as regards the generation of stable or chaotic Malthusian oscillations around the path of balanced growth of the economy.

In a different vein Arthur and McNicoll (1977) considered the issue of the optimal control of the economy in a fully age-structured population from a central planner's perspective. The same authors (1978) also initiated the study of the implications of the dynamics of age structure on the performance of the economy. Their approach has become the standard route to investigate the implication of demographic change, such as the onset of below replacement fertility and ageing in industrialised countries, on economic profiles (Lee, 1994; Lindh and Malmberg, 1999; Miles, 1999; Prskawetz and Fent, 2007). More recently the role of age structure has also been considered within micro-founded growth models, for instance in the 'vintage human capital approach' (Boucekkine et al., 2002), which shows that (exogenous) mortality decline in history could have been a factor of sustained endogenous growth.

This chapter explores the implications of age structure for macroeconomic growth. As a first step we build a general framework embedding the age structure of the population in descriptive growth models in a fully realistic manner. We do this by (1) considering a generic model of descriptive macroeconomic growth and the fundamental equation for the dynamics of a closed, age structured, population, i.e. the Von Foerster partial differential equation (PDE), and (2) integrating them by appropriately modelling the main link function between the economic and demographic systems, i.e. the rate of change of the labour supply. Compared to other works in the literature our approach is fully general. Second, we seek a mathematically convenient reformulation of our model and start to investigate its main dynamic implications, looking at the existence of stationary states, i.e. joint population-economy balanced growth paths, in selected examples. In particular, we consider a basic case of no feedback from the economic system on demographic behaviour, useful to state the demographic assumptions underlying classical exogenous growth models, and a Solow-type model with endogenous fertility modelled as a declining function of per-capita income, as observed during the demographic transition. The model shows the possibility of many equilibria, some of which appear as a consequence of correctly taking age structure into account, i.e. the fact that equilibrium per-capita income is the outcome of both capital dilution and intergenerational transfer effects (Arthur and MacNicoll, 1978). This gives an insight on issues such as the optimum population growth rate, and the overall structure of balanced growth equilibria. Some remarks on the potential of our model for studying the role of age structure for growth are also added.

The chapter is organised as follows. Section 9.2 introduces our general framework. In Section 9.3 we seek a mathematically convenient formulation. In Section 9.4 we consider some selected case-studies. Some conclusions are drawn in Section 9.5.

9.2. A GENERAL FRAMEWORK FOR DESCRIPTIVE MACROECONOMIC GROWTH WITH AGE STRUCTURE

9.2.1. Descriptive Models of Macroeconomic Growth

The most influential descriptive growth schemes are the neoclassical model of Solow (1956) and the growth cycle (Goodwin, 1967). Descriptive models are described by systems of non-linear ODEs of the form

$$\dot{X} = H(X;Z) \tag{9.1}$$

where $X = (x_1, x_2, ..., x_n)$ is the vector of key economic variables (capital per worker, employment, wage share, etc), H a suitable map, and Z a constant vector reflecting the steady input provided by the constant exogenous growth of labour supply and technology. In this paper we assume for simplicity that Z just includes population growth.

Descriptive models show exogenous growth, i.e. growth is promoted by variables such as the total labour supply and the level of technology, which are exogenously defined (exponentially growing over time). In other words, such models explain the mechanics of economic growth provided there is a steady, unexplained, input of population and technology.

Example 1 (Solow's 1956 model without technical progress). In this case X is one-dimensional and is given by the amount of capital per worker $K/L = k$ and H is the map:

$$H(k;Z) = sf(k) - (Z + d)k \tag{9.2}$$

where f is a constant returns to scale neoclassical production function (in intensive form), s the saving ratio, $Z = n$ the growth rate of the labour supply, and d the rate of capital depreciation.

Example 2 (Goodwin's 1967 growth cycle model). X is the 2-dimensional vector (E, V), where E is the employment rate at time t, defined as the ratio between the labour actually employed L and the supply of labour N_s, while $V = w/A$ is the labour share, where w is the real wage and A average labour productivity. The model is

$$\dot{E} = E\left[m(1-V) - \alpha - n_S\right]$$

$$\dot{V} = V\left[-(\alpha + \gamma) + \rho E\right]$$

where $0 < \gamma < \rho$ are the parameters of the 'real wage' Phillips' curve: $w'/w = -(\alpha + \gamma) + \rho E$ governing the relationship between the growth rate of the wage and the employment rate, $m > 0$ is the output–capital ratio, and $\alpha > 0$, $n_s > 0$ respectively denote the rate of change of labour productivity and the labour supply. In this case $Z = (Z_1, Z_2) = (\alpha, n_S)$.

Solow's and Goodwin's model can be integrated to obtain flexible families of growth models: for instance replacing Goodwin's assumption of a Leontief technology with a neoclassical CES production function (Van der Ploeg, 1988) one obtains a family of descriptive neoclassical models with unemployment (Fanti and Manfredi, 2003).

9.2.2. Endogenisation of Age Structure

We start from the fact that under our assumptions the rate of change Z of the labour supply Q:

$$Z = \frac{\dot{Q}(t)}{Q(t)} \tag{9.3}$$

is the only direct 'link function' between the economic sub-system and the demographic one. The labour supply is defined as:

$$Q(t) = \int_{A(X)}^{B(X)} \gamma(a, X) n(a, t) da \tag{9.4}$$

where a denotes an individual's age, $n(a,t)$ is the age density of the population at time t, $\gamma(a, X)$ the participation rate (i.e. the fraction supplying labour at each age), and the interval $(A(x), B(x))$ defines the work age span. The latter is, at least in principle, endogenously determined, as a consequence of workers' choice. The population is assumed to be closed to migration such that its *age density obeys Von Foerster's PDE* (Keyfitz, 1985):

$$\frac{\partial n(a,t)}{\partial a} + \frac{\partial n(a,t)}{\partial t} = -\mu(a,X,N) n(a,t) \tag{9.5}$$

where $\mu(a,X,N)$ is the age-specific mortality rate, which is taken as a function of the economic variables, and of some summary measure of the population state, for instance the total population $N(t) = \int_0^\infty n(a,t) da.$ The PDE (9.5) describes the dynamics of the age density $n(a,t)$ following the ageing process of individuals along their birth cohort over time. Intuitively during each time interval $(t,\ t+h)$ the individual of age a at the beginning of the interval will have age $a+h$ at the end of the interval. In other words, the life course of a given individual is represented in the time-age plane by the 45-degree segments originating on the a-axis at the time of birth of the individual (and obviously ending at his/her death). This means that the natural way to consider the evolution of the age density $n(a,t)$ during $(t,\ t+h)$ is by looking at the variation $\Delta n = n(a+h, t+h) - n(a,t)$. In the absence of migration this variation will only be due to mortality. If the function $n(a,t)$ is regular, Δn can be expanded, for a small enough h, as $(\partial n / \partial a)h + (\partial n / \partial t)h.$ During the same interval some individuals will be removed by mortality, at a rate given by $\mu(a,X,N)n(a,t)h.$ Letting h go to zero finally explains equation (9.5). The PDE (9.5) needs to be completed by a boundary condition, which specifies the rate at which new individuals enter the system at age zero, through births. This is given by the *birth equation*

$$n(0,t)=B(t)=\int_0^\infty \beta(a,X,N)n(a,t)da \tag{9.6}$$

where $B(t)$ is the birth function at time t, and $\beta(a,X,N)$ the age-specific fertility rate. Equations (9.1), (9.3), (9.5), (9.5) and (9.6) define the general form of a descriptive macroeconomic growth model with a fully endogenous age structure, summarised below:

$$\dot{X}=H\left[X(t),Z(t)\right] \tag{9.7a}$$

$$Z(t)=\frac{\dot{Q}(t)}{Q(t)} \tag{9.7b}$$

$$Q(t)=\int_{A(X)}^{B(X)} \gamma(a,X)n(a,t)da \tag{9.7c}$$

$$\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\right)n(a,t)=-\mu(a,X,N)n(a,t) \tag{9.7d}$$

$$n(0,t)=B(t)=\int_0^\infty \beta(a,X,N)n(a,t)da \tag{9.7e}$$

Some remarks are useful to clarify the point. Equations (9.7) show the main ingredients of the problem: (a) the dynamics of the economy, given by equations (9.7a), which depend on the 'link' function $Z(t)$; (b) a definition for the link function, as the rate of change of the labour supply (eq. (9.7b)); (c) a definition for the labour supply (eq. 9.8c); (d) the dynamics of the population (eq. (9.7d)–(9.7e)). In (9.7) the demographic rates $\beta(.),\mu(.)$ and the participation rate $\gamma(.)$ are endogenously determined through possibly non-linear dependencies on their economic inputs. From the viewpoint of non-linear population mathematics, (9.7) is an extension of the classical non-linear age-dependent model first considered by Gurtin and MacCamy (1974). The main difference is that (9.7) includes the external control factors, only implicitly included by Gurtin and MacCamy, in an explicit manner according to an underlying theory of demo-economic interaction. However, Gurtin and MacCamy (1974) were mainly interested in the conditions under which density-dependent conditions were responsible for the onset of a (locally) stable equilibrium age distribution. Here we are mainly interested in the conditions yielding joint persistent solutions of the demographic and economic sub-system (called *stable solutions* in demographic jargon, and 'balanced growth' solutions in the economic jargon), and of course on whether other types of solutions, such as steady demo-economic oscillations, are possible. The dependency of vital rates on the total population N

summarises the possible presence of economic feedbacks not captured by *X*, or of density-dependent effects (see Gurtin and MacCamy, 1974 for the age-structured extension of the logistic equation of classical population mathematics). We further note that the adopted demographic equation is a one-sex equation, as in standard population mathematics. This means that (9.7d) and (9.7e) may be inadequate when differential patterns by gender are assumed.

Initial value problems for (9.7) require initial conditions for the economic variables *X*, and the specification of an initial (i.e. at time $t = 0$) age distribution for the population, of the form $n(a,0) = \psi(a)$, specifying the density of individuals in age group *a* at the initial time.

9.3. A CONVENIENT FORMULATION FOR THE DEMOGRAPHIC COMPONENT

We first look for a convenient formulation of the demographic part of (9.7). The purpose is to simplify the problem by removing components which are unnecessary to understanding the dynamics. We do this in several steps. First we assume that the mortality rate can be written as:

$$\mu(a,X,N) = \mu_1(a) + \mu_2(a,X,N)$$

i.e. as the sum of a 'normal' or baseline component, which is only age-dependent, and of a component also reflecting the action of the economic system. This implies that the overall survival function $\pi(a,t)$, representing the probability of surviving up to any age *a* at time *t*, can be factored as

$$\pi(a,t) = \exp\left[-\int_0^a \mu_1(s)ds\right]\exp\left(-\int_0^a \mu_2\{s,X[t-(a-s)],N[t-(a-s)]\}ds\right) = $$
$$= \pi_1(a)\pi_2(a,t)$$

where $\pi_1(a)$ defines survival to 'normal mortality'. Let us now introduce the following change of variable:

$$u(a,t) = \frac{n(a,t)}{\pi_1(a)} \tag{9.8a}$$

$$q(a,t) = \frac{u(a,t)}{Q(t)} \tag{9.8b}$$

Both $u(a,t)$ and $q(a,t)$ are useful auxiliary quantities to deal with our general framework. The first is defined by the ratio between the absolute (non-normalised) population density and the survival probability to baseline mortality $\pi_1(a)$. Therefore when $\pi_1(a)=\pi(a)$, , i.e. when the overall mortality is only age-dependent, since $n(a,t)=B(t-a)\pi(a)$ holds, then $u(a,t)=B(t-a)$, i.e. it coincides with past births occurring a years ago. In general, $u(a,t)$ represents a population density for which the pure age-dependent component of the risk of mortality has been removed, and only exposure to the economic-dependent component $\mu_2(a,X,N)$ remains, as clear from the first of subsequent equation (9.9).

On the other hand, $q(a,t)$ is obtained by dividing $u(a,t)$ by the total labour supply $Q(t)$. Therefore if we only focus on the work age span (still on the assumption that $\pi_1(a)=\pi(a)$), $q(a,t)$ may be considered the component of the *current total labour force* $Q(t)$ coming from births occurring a years ago, a quantity of strong interest from the demo-economic point of view. Note that $q(a,t)$ is an improper density function.

A partial differentiation shows that $u(a,t)$ and $q(a,t)$ satisfy the following Von Foerster PDEs

$$u_a+u_t=-\mu_2(a,X,N)u(a,t) \qquad (9.9a)$$

$$q_a+q_t=-\left[\mu(a,X,N)+Z(t)\right]q(a,t) \qquad (9.9b)$$

with boundary conditions:

$$u(0,t)=\int_0^\infty m_1(a,X,N)u(a,t)da$$

$$q(0,t)=\int_0^\infty m_1(a,X,N)q(a,t)da$$

where:

$$m_1(a,X,N)=\beta(a,X,N)\pi_1(a)$$

In addition, using (9.8a) the overall labour supply can be rewritten as:

$$Q(t)=\int_{A(X)}^{B(X)} \delta_1(a,X)u(a,t)da$$

where:

$$\delta_1(a,X)=\gamma(a,X)\pi_1(a) \qquad (9.10)$$

Moreover, by (9.8b) one gets the constraint:

$$\int_0^\infty \delta_1(a,X)q(a,t)da=1 \qquad (9.11)$$

Finally, let us seek an expression for the growth rate of the labour supply *Z*. For simplicity we take $A(X)=0, B(X)=\infty$. Indeed, though the ages of entry/exit into/from the labour market are choice variables from the individual's point of view, perhaps bounded by some minimal and maximal 'institutional' age, by suitably endogenising the participation rate at each age $\gamma(a,X)$, essentially any pattern of work participation by age can be described.

By a time differentiation of (9.15), and treating for simplicity *X* as a scalar variable, one finds:

$$\dot{Q}(t)=\dot{X}\int_0^\infty u(a,t)\frac{\partial\delta_1(a,X)}{\partial X}da-\int_0^\infty \delta_1(a,X)u_a(a,t)da-\int_0^\infty \mu_2(a,X,N)\delta_1(a,X)u(a,t)da \tag{9.12}$$

The last expression implies that $Z=\dot{Q}/Q$ can be written as:

$$Z=Z_1+Z_2+Z_3$$

where:

$$Z_1=\dot{X}\int_0^\infty \frac{\partial\delta_1(a,X)}{\partial Y}q(a,t)da$$

$$Z_2=-Q^{-1}\int_0^\infty \delta_1(a,X)u_a(a,t)da$$

$$Z_3=-\int_0^\infty \mu_2(a,X,N)\delta_1(a,X)q(a,t)da$$

Examples can be of help in interpreting the quantities Z_i. The term Z_3 is the (average) component of the mortality rate which is affected by economic inputs. This becomes clear by taking υ_2 independent of age: $\mu_2(a,X,N)=\mu_2(X,N)$. In this case, thanks to (9.11), the following simply holds:

$$Z_3=\int_0^\infty \mu_2(a,X,N)\delta_1(a,X)q(a,t)da=\mu_2(X,N)\int_0^\infty \delta_1(a,X)q(a,t)da=\mu_2(X,N)$$

The term Z_2 depends on the assumptions actually made on the rate of participation to the labour force γ. For instance, if γ is compactly supported over (A,B) and discontinuous at A,B (e.g. when there are minimal/maximal 'institutional' ages of entry and exit into/out of the labour force), we get by a parts integration over age

$$Z_2 = \int_A^B \left[\frac{\partial}{\partial a} \delta_1(a,X) \right] q(a,t)da + \delta_1(A,X)q(A,t) - \delta_1(B,X)u(B,t) \quad (9.13)$$

The latter expression is the balance of the entries in the labour force at the minimal legal age (A), the exits at the maximal age (B), and the relative changes occurring in the participation rate over age. The latter term reflects the compositional effect that arises as the participation rate is not constant over time. If the participation rate is only age-dependent (9.13) reduces to:

$$Z_2(t) = \int_0^\infty \left[\frac{\delta'(a)}{\delta(a)} \right] \delta(a)q(a,t)da + \delta(A)q(A,t) - \delta(B)q(B,t)$$

where the first term is easily seen to be the average of the rate of change of the participation rate over age. Finally the Z_1 term is the component of the overall growth rate of the labour supply due to time changes in its economic determinants X.

To sum up we have written the dynamics of the demographic component in the following manner

$$q_a + q_t = -(\mu_2(a,X,N) + Z(t))q(a,t) \quad (9.14a)$$

$$q(0,t) = \int_0^\infty m_1(a,X,N)q(a,t)da \quad (9.14b)$$

$$Z = \frac{\dot{Q}}{Q} = \dot{X}\int_0^\infty \frac{\partial \delta_1(a,X)}{\partial X} q(a,t)da - Q^{-1}\int_0^\infty \delta_1(a,X)u_a(a,t)da +$$

$$-\int_0^\infty \mu_2(a,X,N)\delta_1(a,X)q(a,t)da \quad (9.14c)$$

plus the constraint (9.10).

The reformulation (9.14) has some advantages. First it uses the 'profile' variable q instead of the absolute age density $n(a,t)$. This has some implications for the search of *stable* (or persistent, or 'balanced') *solutions* for the population. In terms of the absolute profile these are separable solutions of the form $n(a,t) = G(a)e^{rt}$ implying the exponential growth (or decay) of absolute variables (e.g. overall population, labour supply, total births, population in each age group) but unchanging age profiles. In fact, if we look for instance at the profile of the overall population, i.e. the normalised population density $c(a,t) = n(a,t)/N(t)$, [1] the previous position implies $c(a,t) = G(a)/\int_0^\infty G(a)da$. This means that the profile is unchanging over time, i.e. the exponential growth is 'balanced'. Clearly, under (9.14a) such type of solutions correspond to steady states of the profile q. Second, a comparison of (9.14a) with the PDE for the normalised population density $c(a,t)=n(a,t)/N(t)$, given by:

$$c_a + c_t = -\left[\mu_2(a,X,N) + \frac{\dot{N}}{N}\right] c(a,t)$$

indicates that the growth rate of the overall population in the balanced growth state, $r=\dot{N}/N$, must always equal the growth rate Z of the labour supply. Last, 'stable' balanced growth solutions are the only form of stationary solutions of (9.14).

9.4. SOME EXAMPLES

9.4.1. The Case of no Economic Feedback on the demographic System: Asymptotic Balanced Growth of the Economy Forced by Balanced Population Growth

Under constant coefficients

$$\mu(a,X,N)=\pi(a), \beta(a,X,N)=\beta(a), \gamma(a,X)=\gamma(a)$$

no feedback from the economic system is assumed, i.e. vital (mortality and fertility) and participation rates are unaffected by the economic conditions. This case is not economically trivial, as it shows the proper demographic background underlying the standard formulations of descriptive models of economic growth. Moreover, on the applied side, it represents the theoretical framework underlying the common investigations of the impact of changes in the age composition of the population on economic performance under the so-called 'constant vital rates scenario' which are usually supplied by all agencies providing demographic projection (Miles, 1999; Prskawetz and Fent, 2007).

In this case the demographic component is independent of the economic one. Thus we may first characterise the evolution of the demographic component, and then take its output *Z*(t)* to characterise the economic sub-system as follows:

$$\dot{X} = F\left[X(t), Z^*(t)\right]$$

The corresponding long-term dynamics obeys:

$$\dot{X} = F\left[X(t), Z^*_\infty\right]$$

In particular, denoting for brevity $\delta_1(a)=\delta(a)$, the rate of change of the labour supply simplifies to:

$$Z(t)=\int_0^\infty \left[\frac{\delta'(a)}{\delta(a)}\right]\delta(a)q(a,t)da+\delta(A)q(A,t)-\delta(B)q(B,t)$$

The final form for the demographic system is:

$$q_a+q_t+Zq(a,t)=0 \qquad (9.15a)$$

$$q(0,t)=\int_0^\infty m_1(a)q(a,t)da \qquad (9.15b)$$

$$Z(t)=\int_0^\infty \left[\frac{\delta'(a)}{\delta(a)}\right]\delta(a)q(a,t)da+\delta(A)q(A,t)-\delta(B)q(B,t) \qquad (9.15c)$$

plus the constraint (9.12). We now look for stable solutions for the demographic component. As already pointed out, such solutions have the form:

$$n(a,t)=G(a)e^{rt}$$

This implies the following relationships:

$$u(a,t)=\frac{G(a)e^{rt}}{\pi(a)}$$

$$Q(t)=\int_0^\infty \delta_1(a)\frac{G(a)e^{rt}}{\pi(a)}da=Q^*e^{rt}$$

$$Q'(t)=rQ^*e^{rt}$$

$$q(a,t)=\Psi(a)$$

$$q_a+q_t=\Psi'(a)$$

$$Z(t)=Z^*$$

where $\Psi(a)$ is a function of age such that

$$Z^*=\int_0^\infty \left[\frac{\delta'(a)}{\delta(a)}\right]\delta(a)\Psi(a)da+\delta(A)\Psi(A)-\delta(B)\Psi(B)$$

Note in particular that from the definition of Z it follows that its stable form is given by

$$Z^* = \frac{\dot{Q}(t)}{Q(t)} = r$$

which confirms that if the population is stably evolving at rate r then the overall labour supply must evolve at the same rate. In particular, (9.15a) and (9.15b) can be written, using the equality $Z^* = r$, as

$$\Psi'(a) + r\Psi(a) = 0$$

$$\Psi(0) = \int_0^\infty m_1(a)\Psi(a)da$$

This gives:

$$\Psi(a) = \Psi(0)e^{-ra}$$

and therefore

$$\Psi(0) = \int_0^\infty m_1(a)\Psi(0)e^{-ra}da$$

which yields

$$1 = \int_0^\infty m_1(a)e^{-ra}da$$

Recalling that $m_1(a) = \beta(a)\pi_1(a)$ we obtain

$$1 = \int_0^\infty \beta(a)\pi(a)e^{-ra}da$$

The previous expression is a standard Lotka intrinsic equation defining the asymptotic, or 'intrinsic', growth rate of the population in its stable state. Therefore the population eventually achieves a state of stable balanced growth where absolute variables (population size by age group and total, births, overall labour supply, etc.) grow asymptotically at rate $r = Z^*$, keeping an unchanging age profile.

The previous discussion has provided the proof of the following result:

Proposition 1 (case of constant coefficients). If $\mu_2(a,Y) = \mu_2(a)$, $m(a,Y) = m(a)$, $\gamma(a,Y) = \gamma(a)$, then the demographic sub-system decouples. In this case:

All equilibrium solutions of the demographic system are stable solutions, i.e. the demographic system eventually approaches a stable growth path with a speed of growth given by Lotka's intrinsic growth rate *r*. In particular $Z_{\infty}^{*} = r$.

The solution of the economic subsystem, described by the non-autonomous equation: $\dot{X} = H[X, Z(t)]$ converges asymptotically to the solution of the asymptotically autonomous system:

$$\dot{X} = H(X, r)$$

Thus, for instance in the case of Solow's model, the classical Solow's dynamics would arise only asymptotically as a consequence of the stabilisation of the dynamics of the population about its stable state, and it is driven by the following asymptotically autonomous map

$$H_r(k) = sf(k) - (d + r)k$$

On the other hand, the transient phase would be disturbed by the adjustment oscillations of the age distribution, the classical 'stable echoes'.

9.4.2. A Solow-type Model with Endogenous Fertility: Joint Balanced Growth Solutions of the Economy and Population

We now consider an economy governed by the standard Solow equation (9.2) under a constant returns to scale (CRS) Cobb–Douglas production function. As regards the demographic component, we assume that fertility is endogenously related to per capita income $y = Y/N$, where *Y* is absolute output, as largely justified on empirical grounds. Though we will simply take the relation between fertility and income as given, since our interest is in its dynamic implications and not in its microfoundations, it can nonetheless be microfounded both in a traditional, namely Malthusian, perspective, yielding an increasing relation between the fertility rate and income, and in a modern, say Beckerian perspective (Becker and Barro 1988, Becker et al. 1990), where it yields a decreasing relation between fertility and income. Jones (2001) microfounded a humped relation between the fertility rate and per-capita income which explains the increase in fertility sometimes seen at the onset of the demographic transition (Dyson and Murphy 1985).

Here we consider the special but important case: $\beta(a, y) = \beta_1(a)\beta_2(y)$, i.e. changes in per capita income only affect the scale of fertility, represented by the gross reproduction rate $GRR(y) = \beta_2(y)\int_0^{\infty}\beta_1(a)da$ (GRR), but not the age density at childbearing. This is a simplification as we know from empirical studies, in that timing of fertility is related to economic conditions as well. In addition we take only age-dependent mortality and participation. The final form of the system is

$$\dot{k} = sk^{\alpha} - \left(Z(t) + \delta\right)k$$

$$q_a(a,t) + q_t(a,t) = -Z(t)q(a,t)$$

$$q(0,t) = \beta_2(y)\int_0^{\infty} \beta_1(a)\pi(a)q(a,t)\,da$$

$$Z(t) = \frac{\dot{Q}}{Q} = \delta_1(A)q(A,t) - \delta_1(B)q(B,t) + \int_0^{\infty} \delta_1^{'}(a)q(a,t)\,da$$

where

$$y = \frac{Y}{N} = \frac{Y}{L}\frac{L}{N} = y_w \frac{\int_A^B \gamma(a)n(a,t)\,da}{\int_0^{\infty} n(a,t)\,da} = y_w E = k^{\alpha}E$$

where L is labour demand, taken as identically equal to labour supply, y_w the output per worker, and E the ratio between the population actually employed and the total population.

Model (9.16)–(9.17) encompasses many theoretical and simulations models used in the descriptive demo-economic literature. For instance, Solow-type models with endogenous fertility but without age have been used to represent the demographic transition (Strulik, 1997, 1999, 2000; Prskawetz et al., 2000; Fanti and Manfredi, 2003). The above model includes age in such frameworks. On the other hand, model (9.16)–(9.17) represents the full dynamic system underlying the macro-simulative approaches used in the literature on the impact of age structure on macroeconomic performance (Arthur and MacNicoll, 1978; Blanchet, 1988; Miles, 1999; Prskawetz and Fent, 2007).

Let us look at *joint balanced growth solutions of the economy and the population*. By this description we mean solutions characterised by (a) stable exponential growth of all absolute variables (i.e. both demographic, as in the previous example, and economic), and (b) unchanging age profiles of the demographic variable, as a consequence of the position $n(a,t) = G(a)e^{rt}$, and steady state growth of the per-worker economic variable. In particular, since we are considering a Solow model without technical progress, a balanced growth solution of the economy is simply a steady state of the fundamental equation corresponding to a stationary value of Z allowed by an underlying exponential growth of the overall labour supply. From

$$sk^{\alpha} - \left(Z + d\right)k = 0 \qquad (9.16)$$

as $y_w = k^{\alpha}$, we find

$$y_w = \left(\frac{s}{d+Z} \right)^{\frac{\alpha}{1-\alpha}} \tag{9.17}$$

In addition Z must be defined as the intrinsic growth rate of a stably growing population, i.e. $Z = r$ must hold. Setting as before $n(a,t) = G(a)e^{rt}$ we get:

$$y = \left(\frac{s}{r+d} \right)^{\frac{\alpha}{1-\alpha}} E^* \tag{9.18a}$$

$$E^* = \frac{\int_A^B \gamma(a) e^{-ra} \pi(a) da}{\int_0^\infty e^{-ra} \pi(a) da} \tag{9.18b}$$

$$\Psi'(a) = -r\Psi(a) \tag{9.18c}$$

$$\Psi(0) = \beta_2(y) \int_0^\infty \beta_1(a) \pi(a) \Psi(a) da \tag{9.18d}$$

Joint balanced growth solutions, if any, arise from system (9.19), where there is the additional complication that the equilibrium value E^* of E is itself a function of the population growth rate in the balanced growth state. From the two latter equations one finds:

$$1 = \beta_2(y) \int_0^\infty \beta_1(a) \pi(a) e^{-ra} da \tag{9.19}$$

which, using (9.18a) and (9.18b), yields the following 'expanded' intrinsic equation:

$$1 = \beta_2(y(r)) \int_0^\infty \beta_1(a) \pi(a) e^{-ra} da \tag{9.20a}$$

$$y(r) = \left(\frac{s}{r+d} \right)^{\frac{\alpha}{1-\alpha}} E^*(r) = \left(\frac{s}{r+d} \right)^{\frac{\alpha}{1-\alpha}} \frac{\int_A^B \gamma(a) e^{-ra} \pi(a) da}{\int_0^\infty e^{-ra} \pi(a) da} \tag{9.20b}$$

A careful discussion of (9.20) requires that we distinguish among the various possibilities as regards the shape of the $\beta_2(.)$ function, describing how fertility rates by age are scaled by changing levels of income per capita.

If $\beta_2(.)$ is taken to be a monotonic function of income, a simpler discussion of (9.20) can be obtained by setting

$$g(r)=\int_0^\infty \beta_1(a)\pi(a)e^{-ra}da$$

and solving for income per capita:

$$y(r)=\beta_2^{-1}\left(\frac{1}{g(r)}\right) \tag{9.47}$$

In this case the lhs of the previous expression relates possible equilibrium levels of per-capita income to the population growth rate in a stable population. Such a relation has a noteworthy shape due to the interaction between so-called capital dilution (CD) and intergeneration transfer (IT) effects (Arthur and MacNicoll 1978, Blanchet 1988, Manfredi and Fanti 2005). This relation can take, even in the simplistic case of a flat participation rate over (*A,B*), the shape in Fig. 9.1, with very high levels of income per capita when r approaches its lower bound (*–d*), i.e. when the CD effect certainly dominates, then as *r* increases *y(r)* approaches a Worst Population Growth Rate (WPGR), an Optimum Population Growth Rate (WPGR), and finally steadily declines and approaches zero for very large *r*.

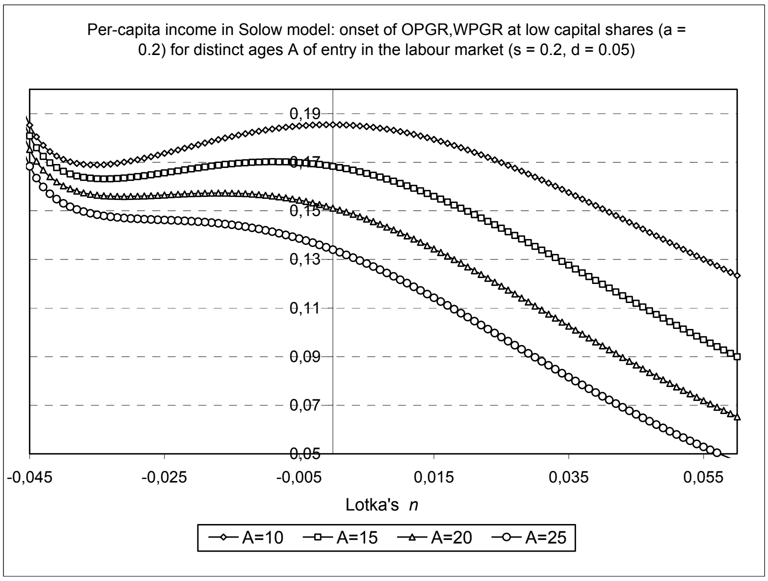

Figure 9.1. Income per capita as a function of the equilibrium population growth rate in a stable population

On the other hand, since $g(r)$ is monotonically decreasing in r, the rhs will be a decreasing function of r when $\beta_2(.)$ is decreasing in income per capita, while it will be increasing in r when $\beta_2(.)$ is increasing. Though computations of the solutions of (9.47) are still to be carried out under realistic parameter values, potentially interesting demo-economic findings are suggested. For instance, consider the case of the decreasing S-shaped $\beta_2(.)$ function, starting from high fertility levels at very low levels of per capita income, then decreasing initially rapidly as income increases, and subsequently slowing down and plateauing at a lower level for very high levels of income per capita. Such a form broadly corresponds to the conventional demographers' description of the demographic transition (Keyfitz, 1985; Chesnais, 1987), and has been used in descriptive economic models of the demographic transition (Strulik, 1997, 1999, 2000; Prskawetz et al., 2000). In this case equation (9.49) may have multiple solutions, each corresponding to a possible state of joint balanced growth of the population and the economy. For example, a frequent case shows five solutions (Fig. 9.2). Though no formal stability analysis was carried out it is tempting to interpret the equilibrium with a high rate of population growth (E_1) as a (locally stable ?) Malthusian trap, the one with the intermediate rate of population change (E_3) as a (locally stable?) post-transitional equilibrium with population growth not too far from stationarity, and finally the equilibrium with very low, largely negative, growth rate (E_5), as a (locally stable?) 'super-modern' equilibrium, a sort of 'richness trap' characterised by sustained below-replacement fertility, as currently spreading in much of the industrialised world (Billari and Kohler, 2004).

The previous results raise interesting considerations as regards the debate on multiple equilibria of the population-economy system in history, the existence and escape from the Malthusian trap, and modern growth regimes.

The possibility of three equilibria (a locally stable Malthusian trap, an intermediate unstable equilibrium and a stable regime of modern growth) is a common result in Solow-type models of the demographic transition (Strulik 1997, 1999, 2000; Fanti and Manfredi 2003). Here it is suggested that the incorporation of age structure might lead to the appearance of two further equilibria, one of which might be interpreted as a 'richness trap' with sustained below-replacement fertility. This phenomenon is the consequence of correctly taking into account the dependence of fertility on per-capita income, as a consequence of the peculiar relation between equilibrium levels of income per capita and regimes of population growth.

The discussion also indicates that once one takes properly into account the overall dynamic interaction between the economic system and the demographic one then: a) the curve *y(r)* will not anymore to be interpreted, as traditionally done, as a locus of equilibria; b) the WPGR and OPGR will not be equilibrium solutions, unless by chance.

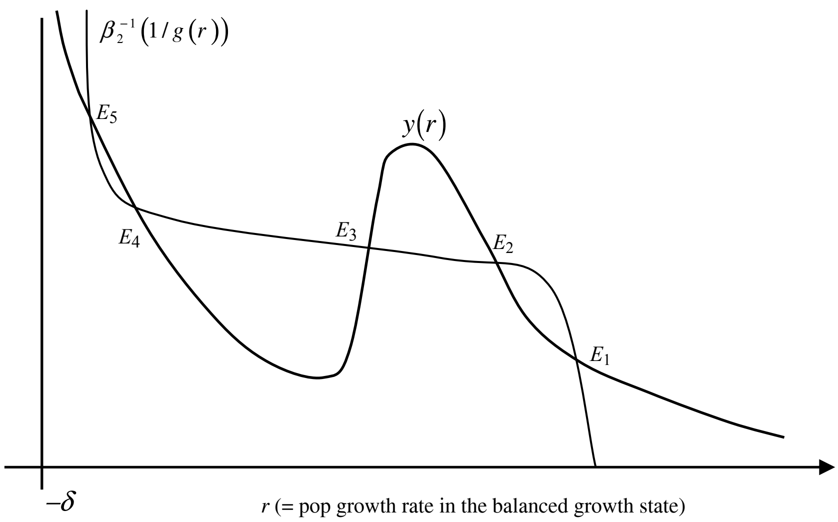

Figure 9.2. From the 'Malthusian' to the 'Richness' trap. Multiple balanced growth equilibria when fertility is a decreasing function of per-capita income according to a 'Demographic Transition' pattern

The Malthusian case can be discussed similarly. In this case at most three equilibria are possible. The case of a humped $\beta_2(.)$ function (Jones 1999) is of course more complicated and leads potentially to a more complex equilibrium structure.

9.4.3. Remarks on Age Structure and Technical Progress

An interesting issue regards our current omission of technological progress. Besides the simple case of exogenous labour-augmenting technical progress, given that our framework focuses on the dynamic role of the population and the labour force, a 'natural' and consistent way to include technical progress would be to endogenise the rate of technical progress $\dot{A}/A$ according to Jones's (1992, 1995) 'idea-based' technical progress, or according to the Boserupian assumption used by Kremer (1993), and extended in many subsequent studies (e.g. Klasen and Nestmann, 2006). These efforts typically postulate: $\dot{A}/A = f_\kappa(P,y)$ where P is the total population, y income per capita, and κ the research productivity per person. Our framework allows us – at least in principle – to go into greater depth in that it models more finely the relation between technological progress and population including age structure. For instance, there are studies documenting that the productivity of researchers is strongly related to the individual's age (Skirbeck, 2003). This implies that if, as commonly taken, the number of innovations per unit time obeys e.g. a Poisson process at each age, then the expected number of innovations in a small interval of time dt might be described as $dt\left[P_1(t)\int_0^\infty g_1(a,t)\xi(a,t)da\right]$ where $P_1(t)$ is the absolute number of individuals working in the research sector at time t, $g_1(a,t)$ is the age density of individuals working in the research sector of the economy, and $\xi(a,t)$ the probability that the single researcher aged a years gives rise to an innovation. If we assume that $P_1(t)=\sigma(t)P(t)$ where $\sigma_1(t)$ is the fraction of the total population working in the research sector we are led to the equation $\dot{A}/A=P_1(t)\left[\sigma_{11}(t)\int_0^\infty g_1(a,t)\xi(a,t)da\right]$, which can be considered an extension of Kremer's (2003) basic equation with a fine specification of the innovation process with respect to age. Many further refinements are of course possible, such as (a) adopting more general specifications, such as Jones's (1992, 1995) idea that the growth rate of technology is also a non-linear function of the level of technology, or that population acts non-linearly on the rate of technological change (Grossman and Helpman, 1991; Aghion and Howitt, 1992), or (b) endogenising the main quantities, e.g. assuming that $\xi=\xi(a,y),\sigma_1=\sigma_1(y)$, i.e. that they are endogenously related to income per-capita.

Obviously the neoclassical descriptive growth perspective considered here – which offers a widely used framework (Grimm and Harttgen, 2008) for investigating the macroeconomic effects of changes in the age structure with the advantage of being well-tailored on classical population mathematics – represents just part of the economic growth story. If our aim is a better understanding of the possible impact of macro-demographic phenomena, such as mortality changes, fertility transitions or population ageing, on the economy, we cannot really neglect the micro-level. Demographic change affects macroeconomic variables, and macroeconomic variables in turn affect individuals' choices, in terms for instance of education, investment in human capital, savings etc. which will ultimately feed back on individuals' demographic choices, and so on. Therefore the reliance on macroeconomic descriptive models is a key limitation of the current paper, which is our aim to remove in future work.

9.5. DISCUSSION

The chapter reports progress in developing a framework fully integrating age structure within descriptive macrodynamic growth models. The ensuing model can be considered the appropriate dynamic framework underlying macro-demographic-macro-simulation models used to investigate the implications of changes in the population's age distribution upon macroeconomic performance. Preliminary analysis of selected subcases, i.e. a Solow model descriptively embedding a fertility transition according to the conventional demographic viewpoint, indicates that the model can provide useful insight on the possible dynamic interaction between population and economic growth. Further mathematical steps will be devoted to a thorough investigation of the steady state structure under realistic parameter constellation, investigation of the stability of balanced growth paths and related oscillation issues. For instance, the inclusion of age structure makes the appearance of sustained Malthusian oscillations around the possible paths of balanced growth a fairly plausible possibility".

From a prospective point of view many other demo-economic problems can be investigated by the present framework such as the inclusion of: a) further endogenous demographic and economic parameters; b) 'not-only scale' effects of the economy on demographic patterns, such as, the shift in childbearing age recently observed in very low-fertility countries (Billari and Kohler, 2004); (c) population and labour force immigration (Manfredi and Valentini, 2000 and references therein) or the marriage process (Billari et al., 2000); (d) other macroeconomic growth environments, such as Goodwin and AK models; (e) further age-related heterogeneity in economic profiles, as the labour productivity across age (Miles, 1999); (f) further issues beyond the growth problem, namely how age structure affects income distribution along growth paths, according to Fanti and Manfredi (2005). This suggests that the present framework is a flexible one deserving further investigation.

APPENDIX

We report some details on the derivation of our framework in an intermediate case in which the vital rates are fully endogenised whereas the parameters of labour supply remain exogenous. In particular, we assume that the participation rate γ is compactly supported on (A, B). Restart from the basic equations

$$Z(t)=\frac{\dot{Q}(t)}{Q(t)}$$

$$Q(t)=\int_A^B \gamma(a)n(a,t)da$$

$$\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\right)n(a,t)=-\mu(a,X,P)n(a,t)$$

$$n(0,t)=B(t)=\int_0^\infty \beta(a,X,P)n(a,t)da$$

We assume that the mortality rate can be written as:

$$\mu(a,Y)=\mu_1(a)+\mu_2(a,X,P) \tag{9A.1}$$

which amounts to stating that mortality is the sum of a 'normal' or baseline component, which is only age-dependent, and of a component which also reflects the action of the inputs from the economic sub-system. This implies that the survival function can be written as

$$\pi(a)=\exp\left[-\int_0^a \mu_1(s)ds\right]\exp\left\{-\int_0^a \mu_2\left[s,X(t-a-s),P(t-a-)\right]ds\right\}=\pi_1(a)\pi_2(a,t)$$

where $\pi_1(a)$ defines survival to 'normal mortality'. Let us introduce the change of variable:

$$u(a,t)=\frac{n(a,t)}{\pi_1(a)}$$

and derive a MacKendrick–Von Foerster PDE for $u(a, t)$. We have:

$$u_a+u_t=\frac{1}{\pi_1(a)}\left[n_a+n_t-n(a,t)\frac{\pi_1'(a)}{\pi_1(a)}\right]$$

i.e.:

$$u_a + u_t = \frac{1}{\pi_1(a)}\left[-\mu_1(a) - \mu_2(a,X,P) - \frac{\pi_1'(a)}{\pi_1(a)}\right]n(a,t)$$

Since:

$$\frac{\pi_1'(a)}{\pi_1(a)} = -\mu_1(a)$$

Then

$$u_a + u_t = -\mu_2(a,X,P)u(a,t)$$

and

$$u(0,t) = \frac{n(0,t)}{\pi_1(0)} = B(t) = \int_0^\infty \beta(a,X,P)\pi_1(a)u(a,t)da$$

In addition:

$$Q(t) = \int_0^\infty \gamma(a)\pi_1(a)u(a,t)da$$

Let us now define:

$$m_1(a,X,P) = \beta(a,X,P)\pi_1(a)$$

$$\delta_1(a,X) = \gamma(a,X)\pi_1(a)$$

Then (9A.1) can be reformulated as

$$Z(t) = \frac{\dot{Q}(t)}{Q(t)}$$

$$Q(t) = \int_0^\infty \delta_1(a)u(a,t)da$$

$$u_a + u_t = -\mu_2(a,Y,P)u(a,t)$$

$$u(0,t) = \int_0^\infty m_1(a,Y,P)u(a,t)da$$

Next let us consider the change of variable:

$$q(a,t) = \frac{u(a,t)}{Q(t)}$$

and let us derive a MacKendrick–Von Foerster PDE for $q(a, t)$. We have:

$$q_a + q_t = \frac{1}{Q(t)}\left[u_a + u_t - u(a,t)\frac{\dot{Q}(t)}{Q(t)}\right]$$

and therefore

$$q_a + q_t = -\left[\mu(a,Y,P) + Z(t)\right]q(a,t)$$

The corresponding boundary condition is:

$$q(0,t) = \frac{u(0,t)}{Q(t)} = \int_0^{\infty} m_1(a,X,P)\frac{u(a,t)}{Q(t)}da = \int_0^{\infty} m_1(a,X,P)q(a,t)da$$

Let us finally seek an equation for *Z*(*t*). The following holds:

$$\dot{Q}(t) = \int_A^B \left\{\delta_1(a)\left[-u_a - \mu_2(a,X,P)u(a,t)\right]\right\}da =$$
$$= (-1)\left[\int_A^B \delta_1(a)u_a da + \int_A^B \mu_2(a,X,P)\delta_1(a)u(a,t)da\right]$$

If $\mu_2(a,X,P)$ is independent of age the previous expression reduces to

$$\dot{Q}(t) = (-1)\left[\int_0^{\infty} \delta_1(a)u_a da + \mu_2(X,P)Q(t)\right]$$

By a parts integration

$$\int_A^B \delta_1(a)u_a da = \left[\delta_1(a)u(a,t)\right]_A^B - \int_A^B \delta_1^{'}(a)u(a,t)da =$$
$$= \left[\delta_1(B)u(B,t) - \delta_1(A)u(A,t)\right] - Q(t)\int_A^B \delta_1^{'}(a)q(a,t)da$$

We thus obtain

$$\dot{Q}(t) = \delta_1(A)Q(t)q(A,t) - \delta_1(B)Q(t)q(B,t) + Q(t)\int_A^B \delta_1^{'}(a)q(a,t)da - \mu_2(X,P)Q(t)$$

and therefore

$$Z(t) = \delta_1(A)q(A,t) - \delta_1(B)q(B,t) - \int_0^{\infty} \delta_1^{'}(a)q(a,t)da - \mu_2(Y,P)$$

which is a general form for the rate of change of the overall labour supply. The final form for the demographic system therefore is:

$$Z(t)=\frac{\dot{Q}}{Q}=\delta_1(A)q(A,t)-\delta_1(B)q(B,t)+\int_0^\infty \delta_1^{'}(a)q(a,t)da-\mu_2(Y,P)$$

$$q_a+q_t=-\left[\mu_2(a,Y,P)+Z(t)\right]q(a,t)$$

$$q(0,t)=\int_0^\infty m_1(a,Y,P)q(a,t)da$$

plus the restraint

$$\int_A^B \delta_1(a)q(a,t)da=1.$$

NOTES

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1. We apologise for the use of the notation *c* ubiquitously used in economics to denote the *consumption* level: unfortunately c is also ubiquitously used in mathematical demography to denote population *composition*. Fortunately no risk of confusion occurs in this chapter since we do not consider consumption.

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